Variational Parabolic Capacity

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(J. Work with T. Kuusi, M. Parviainen)

PDES
Potential theory
Function spaces
In honour of Lars Inge Hedberg (1935-2005)

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p-parabolic equation

$$Hu = u_t - \Delta_p u = 0\,.$$

Definition

A p-superparabolic function u in $E \subset \mathbb{R}^{n+1}$ is a l.s.c. function satisfying the comparison principle on cylinders and it is finite on a dense subset of E.

p-parabolic equation

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Definition

Consider the set $E \subset \mathbb{R}^{n+1}$ and let $A \subset E$

$$R_A^v = \inf\{u : u \geq v \cdot 1_A, u \text{ is superparabolic in } E\}\,,$$

the Réduite or the Reduction of v over A. Usually the function v is a superparabolic function in E, we will mostly be concerned with R_A^1 which can be called the Balayage of A.

Facts about the p-superparabolic functions

1. Locally bounded p-superparabolic functions are weak supersolutions [Kinnunen, Lindqvist, Ann. Math. Pura Appl. -06]

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Facts about the *p*-superparabolic functions

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2. weak supersolutions have a l.s.c. representative which is a *p*-superparabolic function [Kuusi, Differential Integral Equations -09]
3. for each Radon measure μ there is a potential u_μ which is a *p*-parabolic function

   $$(u_\mu)_t - \Delta_p u_\mu = \mu$$

   in the weak sense. [Kinnunen, Lukkari, Parviainen, JFA -10, J. Fixed Point Theory Appl. -13]

Different notions of capacity

Let our domain be the open set $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$:

Using the capacitary potential charge

$$C_0(K) = \mu_{R_K}(K) \tag{1}$$

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Let our domain be the open set $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$:

Using the maximal charge

$$C_1(E) = \sup\{\mu(\Omega_T), 0 \le u_\mu \le 1, \text{ in } \Omega_T, \mathrm{supp}\mu \subset E\} \tag{2}$$

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Let our domain be the open set $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$:

For $p = 2$ using the function space (variational) approach

$$W = \{v \in L^2(0, T; H_0^1(\Omega)); v_t \in L^2(0, T; H^{-1}(\Omega))\}$$

Smooth functions are dense in W, and we can thus define a "capacity"

$$C_{var}(K, \Omega_T) = \inf\{\|u\|^2_{W(\Omega_T)} : u \geq 1_K; u \in C_0^\infty(\Omega \times \mathbb{R})\} \tag{3}$$

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We can extend as

$$C_{var}(E, \Omega_T) = \inf\left\{\sup\{C_{var}(K); K \subset O,\ K \text{ compact}\}; O \supset E;\ O \text{ open set}\right\}$$

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Different notions of capacity

The capacitary potential charge gives the same capacity as the maximal charge on compact sets i.e.

Theorem (Kinnunen, Korte, Kuusi, Parviainen, Math. Ann. -13)

$$C_0(K) = C_1(K)\,. \tag{4}$$

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What about the variational "capacity"?

Theorem (Pierre, SIAM -83)

Let $p = 2$ then

$$C_{var}(K) \approx C(K)\,.$$

What about $p > 2$?

It was proposed by Pierre, SIAM -83 that

$$W = \{v \in L^p(0,T;W_0^{1,p}(\Omega)); v_t \in (L^p(0,T;W_0^{1,p}(\Omega)))'\}$$

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$$W = \{v \in L^p(0,T;W_0^{1,p}(\Omega)); v_t \in (L^p(0,T;W_0^{1,p}(\Omega)))'\}$$

and define the variational capacity as

$$C_P(K,\Omega_T) = \inf\{\|u\|_{W(\Omega_T)} : u \geq 1_K; u \in C^\infty(\Omega \times \mathbb{R})\} \tag{5}$$

This capacity has been studied in a paper by Droniou, Poretta, Prignet, Pot. Anal. -03. Where the studied the homogeneous Dirichlet measure data problem with L^1 initial data, and showed that there is a unique renormalized solution if the measure does not charge a set of zero capacity.

What about $p > 2$?

It turns out that

$$C(K) \not\approx C_P(K, \Omega_T)^q$$

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Define

$$V = L^p(0, T; W_0^{1,p}(\Omega));$$
$$W = \{u \in V; u_t \in V'\},$$

The norm in the space W does not scale well under intrinsic rescaling, this comes from the dual powers p and p', thus in order to remedy this we need to consider

$$\|u\|_{V(\Omega_T)}^p + \|u_t\|_{V'(\Omega_T)}^{p'}$$

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This quantity scales quadradically with respect to intrinsic rescaling $\lambda^{-1}u(x, \lambda^{2-p}t)$, which is the same behavior as the energy of a solution to our equation

$$u_t - \Delta_p u = 0\,.$$

Intrinsic variational capacity for $p > 2$

With all this in mind we will define an intrinsic variational capacity as

$$C_{var}(K, \Omega_T) = \inf\{\lambda^2 : \lambda^2 = \|v\|^p_{V(\Omega_{\lambda^{2-p}T})} + \|v_t\|^{p'}_{V'(\Omega_{\lambda^{2-p}T})}, \\ v \geq 1_K; v \in C^\infty(\Omega \times \mathbb{R})\}$$

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Some properties of C_{var}

Let $K \subset \Omega_T$ be a compact set

1. $C_{var}(K, \Omega_T) < \infty$
2. $C_{var}(K_1, \Omega_T) \leq C_{var}(K_2, \Omega_T)$
3. $C_{var}(K, \Omega_{T_1}) \leq C_{var}(K, \Omega_{T_2})$ for $T_1 \leq T_2$.
4. Let K_i, $i = 1, 2, \ldots$ be compact sets in Ω_T then

$$\lim_{i \to \infty} C_{var}(K_i, \Omega_T) = C_{var}(\cap_i K_i, \Omega_T)\,.$$

Global version

Lemma (B.A., Kuusi, Parviainen, DCDS -15)

Define

$$C_{var}(K,\Omega_\infty) = \inf\{\|v\|^p_{V(\Omega_\infty)} + \|v_t\|^{p'}_{V'(\Omega_\infty)}, v \geq 1_K; v \in C^\infty(\Omega \times \mathbb{R})\}$$

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Comparison with C_P

The capacity defined in [DPP] w.r.t. to the norm of the space W is related to the variational capacity for $T = \infty$

$$\min\{C_P(K,\Omega_\infty)^p, C_P(K,\Omega_\infty)^{p'}\} \leq C_{var}(K,\Omega_\infty) \leq \max\{C_P(K,\Omega_\infty)^p, C_P(K,\Omega_\infty)^{p'}\}$$

In particular, zero sets coincide.

This is all nice, but what can we do with it?

Define the energy of a p-superparabolic function as

$$\|u\|_{en} = \sup_{0<t<T} \frac{1}{2}\int_\Omega u^2(x,t)dx + \int_0^T \int_\Omega |Du|^p dxdt$$

and define the capacity

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Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let K be a finite union of closed space-time cylinders, set $\lambda^2 = C_{var}(K, \Omega_T)$, and suppose that $K \subset \Omega_{\lambda^{2-p}T}$. Then

$$C_{var}(K, \Omega_T) \approx C_{en}(K, \Omega_{\lambda^{2-p}T})$$

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Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let K be a finite union of closed space-time cylinders, and suppose that $K \subset \Omega_T$. Then

$$C(K,\Omega_T) \approx C_{en}(K,\Omega_T)$$

Taking a limit!

Note that C_{var} and C_P are stable w.r.t. to decreasing limits of compact sets, but it is unknown whether C_{en} is! This gives our limit result

Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let K be a compact set, let $\lambda^2 = C_{var}(K, \Omega_T)$, and suppose that $K \subset \Omega_{\lambda^{2-p}T}$. Then

$$C_{var}(K, \Omega_T) \approx C(K, \Omega_{\lambda^{2-p}T}) .$$

C does not depend on T, thus

$$C_{var}(K, \Omega_T) \approx C(K, \Omega_\infty) .$$

Taking a second limit!

Our previous stability result for C_{var} w.r.t. to T gives

Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let K be a compact set in Ω_∞ then

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Reminder

$$\begin{aligned} C_{var}(K,\Omega_\infty) &= \inf\{\|v\|^p_{V(\Omega_\infty)} + \|v_t\|^{p'}_{V'(\Omega_\infty)}, v \geq 1_K; v \in C^\infty(\Omega \times \mathbb{R})\} \\ C(K,\Omega_\infty) &= \sup\{\mu(\Omega_\infty), 0 \leq u_\mu \leq 1, \text{ in } \Omega_\infty \text{ and } \operatorname{supp}\mu \subset K\} \end{aligned} \tag{6}$$

Estimates of simple sets

Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let $\varphi : [t_1, t_2] \to \Omega$ *be a Lipschitz curve and let* $K \subset \Omega$ *be a set with elliptic p-capacity* 0*, then the set*

$$K_\varphi = \{(x + \varphi(t), t) : x \in K, t \in [t_1, t_2]\}$$

has parabolic capacity 0*.*

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Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let $K \subset \Omega_\infty$ *be a compact set, then*

$$\int_0^\infty cap_p(\pi_t(K), \Omega) dt \leq C_{var}(K, \Omega_\infty)$$

where $\pi_t(K) = \{x : (x, t) \in K\}$*.*

Estimates of simple sets

Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let $Q_r = B(x_0, r) \times (t_0 - \tau, t_0)$ *where* $\tau < t_0$ *and* $Q_{2r} \subset \Omega_T$*, then for* $2 \leq p < n$

$$C(\overline{Q}_r, \Omega_\infty) \approx r^n + \tau r^{n-p}$$

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Theorem (B.A., Kuusi, Parviainen, DCDS -15)

Let $Q_r^+ = B(x_0, r) \times (t_0, t_0 + \tau)$ *be such that* $Q_{2r} \subset \Omega_\infty$ *and let*

$$\mathcal{H} = \{(x, h(x)) : x \in \overline{B}(x_0, r)\}$$

where $h \in C(\mathbb{R}^n)$ *satisfies* $h(x) = t_0$ *on* $\partial B(x_0, r)$ *and* $\mathcal{H} \subset Q_r^+$*. Then*

$$c^{-1} \left(\int_0^\infty cap_p(\pi_t(\mathcal{H}), \Omega)\, dt + r^n \right) \leq C(\mathcal{H}, \Omega_\infty) \leq c(r^n + \tau\, r^{n-p})$$

with $c = c(n, p)$.

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